

Perfect Actions with Chemical Potential *

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Preprint HLRZ 1997-71

February 1, 2008

Abstract

We show how to include a chemical potential μ in perfect lattice actions. It turns out that the standard procedure of multiplying the quark fields $\Psi, \bar{\Psi}$ at Euclidean time t by $\exp(\pm\mu t)$, respectively, is perfect. As an example, the case of free fermions with chemical potential is worked out explicitly. Even after truncation, cut-off effects in the pressure and the baryon density are small. Using a (quasi-)perfect action, numerical QCD simulations for non-zero chemical potential become more powerful, because coarse lattices are sufficient for extracting continuum physics.

*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FC02-94ER40818.

Understanding strongly interacting matter at *finite baryon density* is a long-standing and challenging problem, motivated for instance by relativistic heavy ion collision and by the physics of neutron stars. The standard procedure to formulate lattice QCD at a finite chemical potential μ includes a factor $\exp(\pm\mu)$ in the time-like link variables [1]. As a consequence, the Euclidean action is complex, the Boltzmann factor cannot be interpreted as a probability, and standard Monte Carlo techniques fail.

The usual method to handle a chemical potential is to simulate at $\mu = 0$, and to include the baryon number term by some re-weighting technique in measured observables [2]. However, this method is tractable only on small physical volumes V , for a recent review see Ref. [3]. The essential numerical problem is to measure exponentially suppressed observables, like the partition function ratio $Z(\mu)/Z(0) \sim \exp(-\beta V[f(\mu) - f(0)])$. Here β is the inverse temperature and $f(\mu)$ is the free energy density. In numerical simulations the above ratio arises as an average over many positive and negative contributions. Hence its accurate determination requires tremendous statistics. An improved lattice action can not directly solve this sign problem, but it would help because it suppresses the artifacts due to the finite lattice spacing.

As a particularly troublesome effect caused by lattice artifacts, there is an upper limit for the possible fermion number density on the lattice. The value of this limit depends on the lattice action. It can be understood from the maximal coupling distance in the time-like direction, which restricts the exponential growth of the leading term at $\mu \rightarrow \infty$. An improved action can weaken this unphysical saturation effect, and push it to a larger chemical potential.

As a further problem, chiral symmetry is already restored — and a non-zero quark number sets in — as μ reaches half the pion mass [4]. Generally the quenched approximation has been blamed for that threshold [5], but — at least with respect to the quark number — it persists in simulations using the Glasgow method [6]. It could be a further manifestation of lattice artifacts, which might cause a spurious scattering of the eigenvalues [7]. This problem further motivates the use of improved lattice actions at finite baryon density, since they render the observables more continuum like than a standard lattice formulation.

Perfect lattice actions provide a possibility to completely eliminate cut-off effects. In practice, approximately perfect actions have been constructed, and they have led to a significant reduction of finite lattice spacing artifacts [8, 9, 10, 11]. When applied to a perfect action at $\mu = 0$, the standard procedure of multiplying the fermion fields $\Psi, \bar{\Psi}$ at time t by $\exp(\pm\mu t)$, respectively, turns out to be quantum perfect in the fully interacting theory. We use the example of a free quark to demonstrate that — even after truncation — the resulting lattice action approximates continuum physics very well.¹ On coarse lattices, the scaling behavior of observables is strongly improved

¹Recently another paper has appeared, where a chemical potential was included in a (classically)

compared to the standard action and other proposals. For discussions of the free standard lattice fermions at $\mu > 0$, see the second Ref. in [1] and Refs. [4, 12].

Let us consider a block factor n renormalization group transformation, which maps a fine lattice theory of link variables U and fermion fields $\Psi, \bar{\Psi}$ onto a coarse lattice theory with fields $U', \Psi', \bar{\Psi}'$. The actions $S[U, \Psi, \bar{\Psi}]$ and $S'[U', \Psi', \bar{\Psi}']$ on the fine and coarse lattice at $\mu = 0$ are related by

$$\exp(-S'[U', \Psi', \bar{\Psi}']) = \int \mathcal{D}U \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp\left(-S[U, \Psi, \bar{\Psi}] - T[U', \Psi', \bar{\Psi}', U, \Psi, \bar{\Psi}]\right), \quad (1)$$

where T characterizes the renormalization group transformation. It must be chosen such that the partition function remains invariant, $Z' = Z$. Starting on a very fine lattice with quark mass $m/(nN)$ and inverse temperature βnN , and performing N renormalization group steps with block factor n , leads in the limit $nN \rightarrow \infty$ to a perfect action $S^*[U, \Psi, \bar{\Psi}]$ for quarks of mass m at inverse temperature β .

Now we address the question how to incorporate the chemical potential. In the continuum this can be done simply by a substitution in momentum space,

$$\Psi(k) \rightarrow {}^\mu\Psi(\vec{k}, k_4 - i\mu), \quad \bar{\Psi}(k) \rightarrow {}^\mu\bar{\Psi}(\vec{k}, k_4 - i\mu). \quad (2)$$

In perturbation theory, it is equivalent to leaving the fermionic fields unchanged and replacing

$$k_4 \rightarrow k_4 + i\mu \quad (3)$$

in the free propagator instead. A priori it is not obvious how to transfer this substitution to a standard lattice formulation. An early guess replaced $\sin k_4 \rightarrow \sin k_4 + i\mu$ in the fermionic propagator, but this formulation does not have a proper continuum limit. Instead it turned out that one should use exactly the same substitution (2) resp. (3) also on the lattice to incorporate μ in the actions resp. propagators of Wilson or staggered fermions [1].

Hence, on the initial fine lattice a chemical potential μ can be introduced by replacing $\Psi(\vec{x}, t)$ and $\bar{\Psi}(\vec{x}, t)$ ($t \in [0, \beta]$) in the Wilson action $S[U, \Psi, \bar{\Psi}]$ by

$${}^\mu\Psi(\vec{x}, t) = \exp(\mu t)\Psi(\vec{x}, t), \quad {}^\mu\bar{\Psi}(\vec{x}, t) = \exp(-\mu t)\bar{\Psi}(\vec{x}, t). \quad (4)$$

The chemical potential appears as a purely imaginary constant Abelian gauge potential $A_4 = i\mu$. Thus, the fields ${}^\mu\Psi(\vec{x}, t)$ and ${}^\mu\bar{\Psi}(\vec{x}, t)$ can be viewed as being parallel transported to $t = 0$. This ensures covariance under the corresponding Abelian gauge transformations. In the presence of a chemical potential, the renormalization group transformation must be modified if one wants to manifestly preserve this additional symmetry. This is achieved naturally by using $T[U', {}^{\mu n}\Psi', {}^{\mu n}\bar{\Psi}', U, {}^\mu\Psi, {}^\mu\bar{\Psi}]$ with

$${}^{\mu n}\Psi'(\vec{x}', t') = \exp(\mu n t')\Psi'(\vec{x}', t'), \quad {}^{\mu n}\bar{\Psi}'(\vec{x}', t') = \exp(-\mu n t')\bar{\Psi}'(\vec{x}', t'). \quad (5)$$

perfect action of the 2d $O(3)$ model [13].

Note that on the coarse lattice $t' \in [0, \beta/n]$. Performing the corresponding renormalization group transformation results in the action $S'[U', {}^{\mu n} \Psi', {}^{\mu n} \bar{\Psi}']$. Indeed, this is exactly what one obtains by applying the standard procedure to include a chemical potential μn directly in the coarse lattice action S' . Starting with a small chemical potential $\mu/(nN)$, and again iterating the blocking procedure N times, yields the perfect action $S^*[U, {}^\mu \Psi, {}^\mu \bar{\Psi}]$ with chemical potential μ in the limit $nN \rightarrow \infty$. Hence, performing the standard procedure of including μ in a perfect action — constructed at $\mu = 0$ — yields a perfect action at arbitrary μ . We emphasize that this argument applies to the fully interacting quantum theory.

As a special case, in perturbation theory one may send the blocking factor $n \rightarrow \infty$. Then we don't need to iterate any more; the perfect action is obtained for $N = 1$. We call this technique “blocking from the continuum” [10], because one does not start the blocking process from a fine lattice but directly from the continuum theory. From that procedure it is particularly evident that the continuum relation (3) is inherited without alteration by the perfect lattice propagator. This agrees with the result that we derived above using a finite blocking factor.

For a free fermion with mass m at $\mu = 0$, a perfect action has been constructed in Ref. [10],

$$\begin{aligned} S^*[\Psi, \bar{\Psi}] &= \int_B \frac{d^4 k}{(2\pi)^4} \bar{\Psi}(-k) \Delta^{-1}(k) \Psi(k), \\ \Delta(k) &= \sum_{l \in \mathbf{Z}^4} \frac{\Pi^2(k + 2\pi l)}{i\gamma_\mu(k_\mu + 2\pi l_\mu) + m} + \frac{1}{\alpha}, \\ \Pi(k) &= \prod_\mu \frac{\hat{k}_\mu}{k_\mu}, \quad \hat{k}_\mu = 2 \sin \frac{k_\mu}{2}. \end{aligned} \tag{6}$$

Here, $B =]-\pi, \pi]^4$ is the Brillouin zone and α is a parameter in the renormalization group transformation. At $\alpha = \infty$ and $m = 0$, the perfect action is chirally invariant, and hence — in agreement with the Nielsen-Ninomiya theorem — the action is nonlocal [14]. At finite α , on the other hand, chiral symmetry is explicitly broken in the action — though still present in the observables [15] — and the action becomes local. In particular, for $\alpha = (e^m - m - 1)/m^2$, locality — in the sense of an exponential decay of the couplings — is optimal.

Applying the standard method of including the chemical potential, the perfect action at $\mu > 0$ takes the form $S^*[\mu \Psi, {}^\mu \bar{\Psi}]$. In momentum space we obtain

$$S^*[\mu \Psi, {}^\mu \bar{\Psi}] = \int_B \frac{d^4 k}{(2\pi)^4} \bar{\Psi}(-k) \Delta^{-1}(\vec{k}, k_4 + i\mu) \Psi(k). \tag{7}$$

Let us consider the cut-off effects in the pressure p and the baryon number density n_B on the lattice. In the continuum, for massless quarks at zero temperature, these

quantities are given by

$$p = \frac{\mu^4}{6\pi^2} \quad , \quad n_B = \frac{2\mu^3}{9\pi^2} . \quad (8)$$

On the lattice they take the form

$$\begin{aligned} p &= \int_B \frac{d^4 k}{(2\pi)^4} \left[\log \det \Delta(\vec{k}, k_4) - \log \det \Delta(\vec{k}, k_4 + i\mu) \right], \\ n_B &= \frac{1}{3} \frac{\partial}{\partial \mu} p . \end{aligned} \quad (9)$$

The lattice propagator for a perfect action is given in eq. (6), but one may also insert other fermionic lattice propagators known from the literature, such as the Wilson fermion, staggered fermions or the so-called D234 action [16]. The latter is an extension of the Wilson fermion in the spirit of Symanzik’s improvement program: the artifacts due to the finite lattice spacing are cancelled in the leading order by additional couplings along the axes.

In order to make the perfect action applicable, its couplings have to be truncated to a short range. Hence a fair comparison to other lattice fermions can only deal with a truncated perfect fermion. In Ref. [11] we performed such a truncation for the perfect fermion action (6) — with the parameter α optimizing locality — to the couplings in a unit hypercube. Figure 1 illustrates the cut-off effects in the ratio p/μ^4 as a function of μ in lattice units for the zoo of lattice fermions mentioned above. In the continuum limit $\mu \rightarrow 0$, they all approach the correct value $1/6\pi^2$. At finite μ , the Wilson action suffers from a severe cut-off dependence. The behavior is somewhat better for staggered fermions, but the artifacts are still quite bad. The D234 action is successful at small μ , but around $\mu = 1.5$ it collapses completely. The action of the “hypercube fermion” (truncated perfect fermion), however, remains close to the continuum value of the considered ratio over a wide range of chemical potentials. For example, the considered ratio deviates by less than 50% from the continuum value up to $\mu = 0.8$ for the Wilson and the staggered fermion, up to $\mu = 1.6$ for the D234 action, and up to $\mu = 8.2$ for the hypercube fermion. We note that an alternative truncated fixed point fermion, which has been constructed by the Bern/Boulder group [17], shows a similar scaling quality; it stays even a little closer to the continuum value than the “hypercube fermion” in Figure 1.

Figure 2 illustrates the behavior of the baryon number density n_B divided by μ^3 . Again we compare the Wilson fermion, the staggered fermion, the D234 action and the truncated perfect action. We see that the behavior of our second scaling quantity, n_B/μ^3 , is qualitatively very similar to p/μ^4 . We summarize the outcome of our comparison again: the Wilson and staggered fermion are plagued by large artifacts even at small μ , the D234 action is good on fine lattices but disastrous on very coarse ones, and the “hypercube fermion” stays by far closest to the continuum value. As a further example, the baryon number susceptibility $\chi_B = \partial n_B / \partial \mu$ leads to a scaling quantity χ_B/μ^2 , which again behaves similarly for the lattice fermions

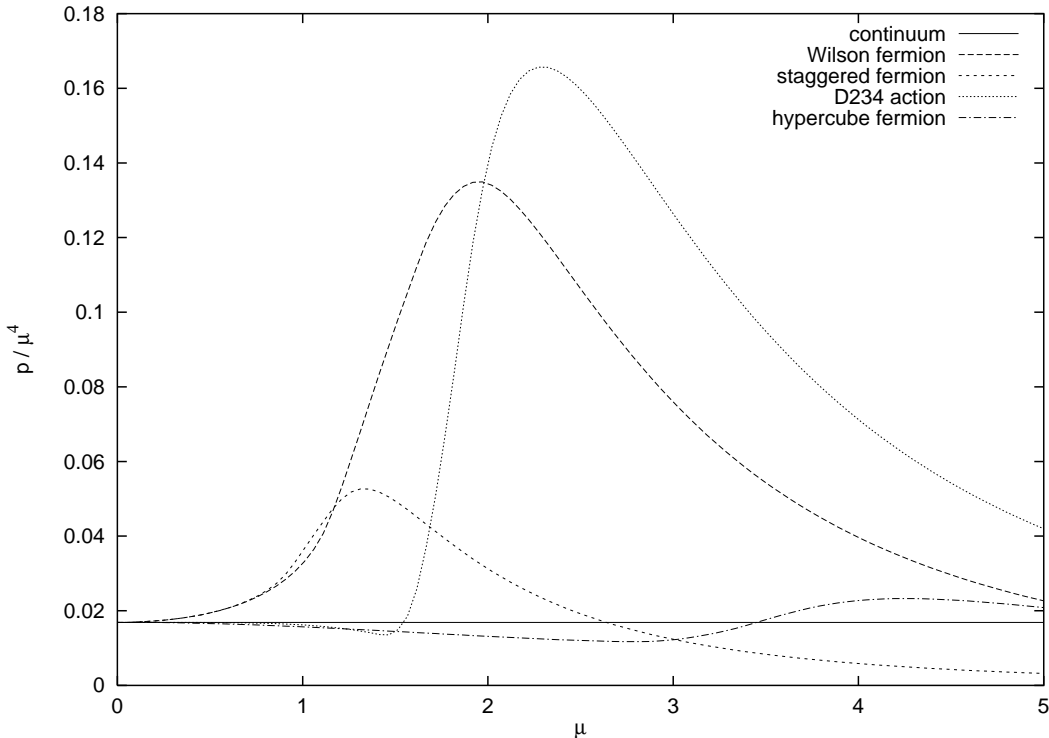


Figure 1: *The ratio p/μ^4 as a function of the chemical potential μ in lattice units. The solid line is the continuum value $1/6\pi^2$. The dashed lines corresponds to the Wilson action ($r = 1$), the staggered fermion action, the D234 action and the truncated perfect action (“hypercube fermion”).*

considered above. This is also in qualitative agreement with comparisons done for other quantities before: in Ref. [11] we compared another thermodynamic scaling ratio, p/T^4 at $\mu = 0$, as well as the dispersion relations, and we found in both cases qualitatively the same behavior as in Figures 1 and 2.

Figure 3 illustrates the saturation effect at large chemical potential for various lattice actions. At large μ , most lattice actions lead to an asymptotic fermion number density $n_f = 32$. That number corresponds to an occupation of each lattice point by 16 flavors with spin up and down. This also includes the doubler fermions, which are not suppressed any longer at sufficiently large μ . The Wilson action with Wilson parameter $r = 1$ is special, because in that case the time-like doublers are completely removed from the spectrum (their mass diverges) [18]. As a consequence, the asymptotic value for n_f is reduced to 16. For the staggered fermions the doubling is reduced by construction, hence their saturation level is only 8. It should be noted that a non-truncated perfect action does not suffer from any saturation artifacts; it behaves like the continuum action. This is due to the presence of couplings over infinite distances in Euclidean time, even though these couplings are exponentially small. In general, the maximal saturation level is proportional to the

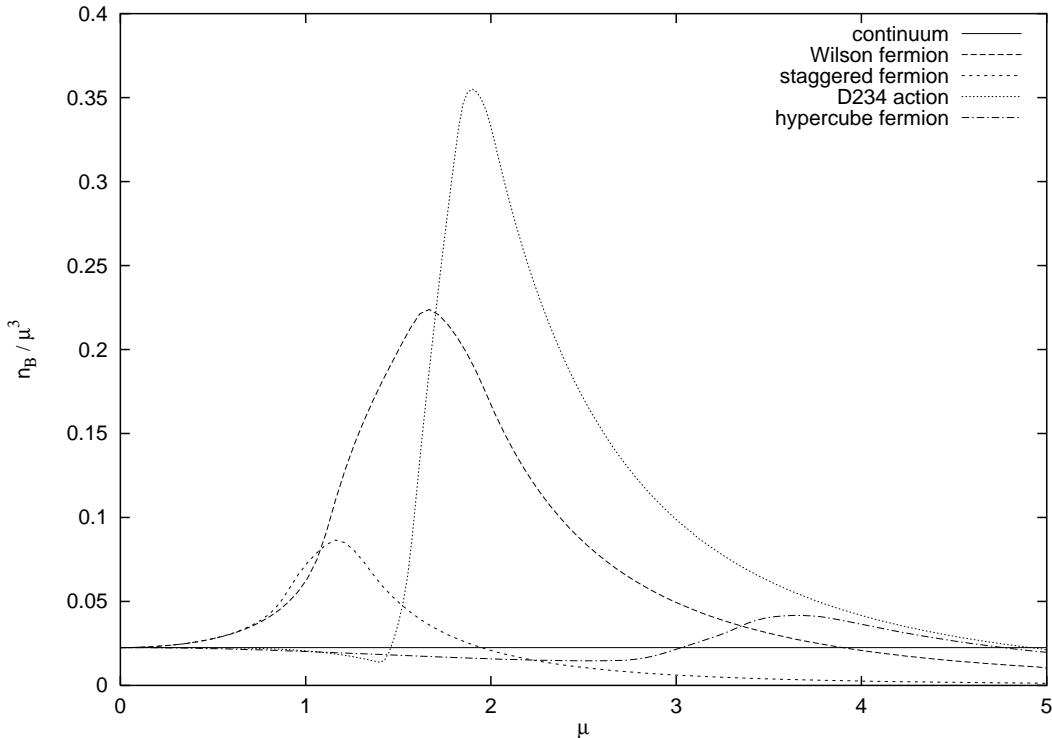


Figure 2: The ratio n_B/μ^3 as a function of the chemical potential μ in lattice units. The solid line is the continuum result $2/9\pi^2$. The dashed lines corresponds to the Wilson action ($r = 1$), the staggered fermion action, the D234 action and the truncated perfect action (“hypercube fermion”).

largest time separation with non-zero couplings.² Since the hypercube fermion has been truncated at time separation 1, it saturates at $n_f = 32$. From this point of view, the D234 action could, in principle, reach $n_f = 64$ because it couples over time separation 2. However, similar to the effect described above for the Wilson action with $r = 1$, the actual saturation level is only half of that.³

Figure 3 further reveals that, as μ grows, the saturation occurs soon for the $r = 1$ Wilson fermion, and at about the same point for the D234 fermion. Staggered fermions saturate at even smaller μ . The saturation can be delayed, however, for Wilson fermions with r close to (but not equal to) 1, which generates a very heavy class of doublers. Then n_f reaches a first plateau at 16, performs another jump as μ catches up with the heaviest doubler mass, and finally saturates at $n_f = 32$. Also for the hypercube fermion the saturation is significantly delayed, and again we recognize a jump as μ exceeds the heaviest doubler mass, so that the final saturation

²This provides an argument that any (exactly) perfect action (in more than one dimension) must couple over infinite distances.

³What happens technically is that the apparently dominant term at $\mu \rightarrow \infty$ cancels for the Wilson fermion at $r = 1$ and for the D234 action.

sets in. And in this case, in contrast to the Wilson fermion with $r \approx 1$, the delayed saturation goes along with an improved behavior at small μ .

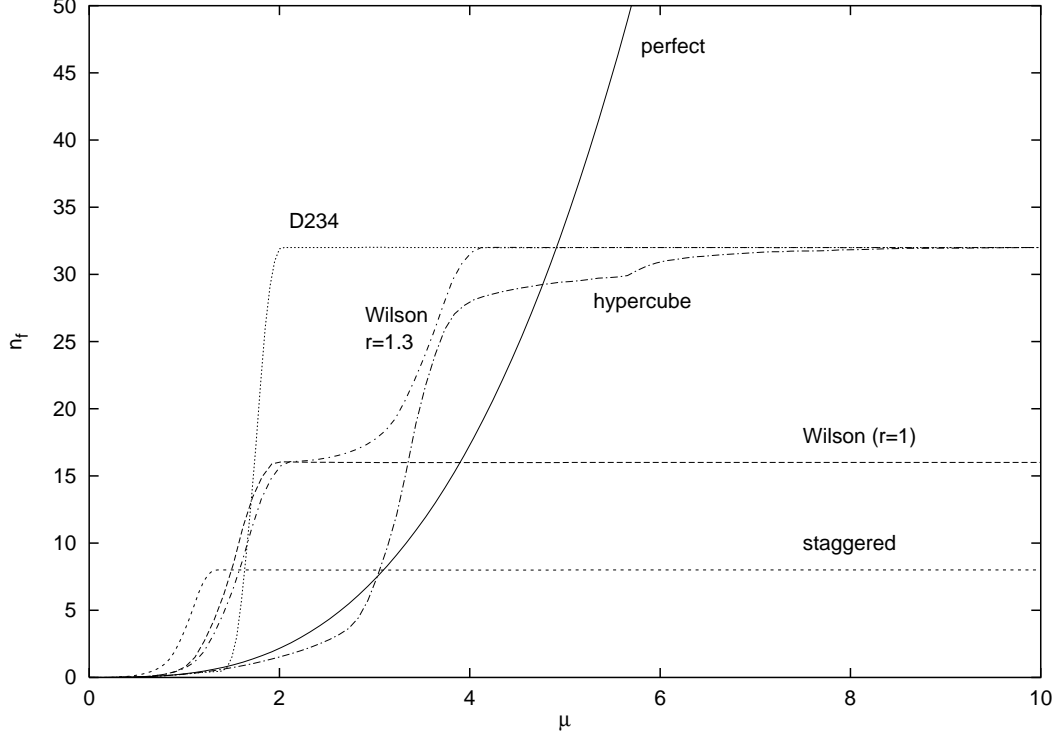


Figure 3: *The fermion number density n_f as a function of μ for various lattice fermions. The Wilson action with $r \neq 1$, the D234 action and the hypercube fermion saturate at $n_f = 32$, whereas the Wilson fermion with $r = 1$ (the staggered fermion) only reaches $n_f = 16$ ($n_f = 8$). Compared to the usual fermion actions, the saturation is delayed significantly for the hypercube fermion. For the perfect fermion action, n_f is not bounded.*

We have shown how to *incorporate a chemical potential in a perfect lattice action* for full QCD. In an application to free fermions we observed a *strongly improved scaling behavior* at finite μ , even after truncation. We further confirm the scenario that the unphysical saturation effect is tamed for a (truncated) perfect action. It is straightforward to extend this study to more general cases, involving for instance a finite quark mass ⁴, a finite temperature or an anisotropic lattice. The improvement permits lattice simulations for QCD at finite baryon density to be performed on much coarser lattices than it is the case for a standard lattice action. Although there is also the major sign problem due to the complex action, such an improvement is important.

⁴A large fermion mass could be of interest along the lines of the static quark method [19].

Part of this work has been done at the University of Erlangen. We thank for its hospitality. We also thank M.-P. Lombardo for reading the manuscript and for very helpful comments, and F. Karsch for useful discussions. Finally, we have benefited from our collaboration with R. Brower, S. Chandrasekharan and K. Orginos.

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